JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 4, November 2012

ROBUSTLY TRANSITIVE SETS WITH SHADOWING

Daejung Kim*

Abstract. Let f be a diffeomorphism of a closed C^∞ manifold M. We show that C^1-generically, if f has the shadowing property on a robustly transitive set, then it is hyperbolic.

1. Introduction

Let M be a closed C^∞ manifold, and denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Denote by $\mathrm{Diff}(M)$ the space of diffeomorphisms of M endowed with the C^1-topology. Let $f \in \mathrm{Diff}(M)$, and let $P(f)$ be the set of periodic points of f. Let Λ be a closed f-invariant set. Denote by $f|_\Lambda$ the restriction of f to a subset Λ of M. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b} \subset M$ $(-\infty \le a < b \le \infty)$ is called a *δ-pseudo orbit* of $f \in \mathrm{Diff}(M)$ if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$.

We say that Λ is *shadowing* for f (or $f|_\Lambda$ has the *shadowing property*) if for every $\epsilon > 0$, there is a $\delta > 0$ such that for any δ-pseudo orbits $\{x_i\}_{i\in\mathbb{Z}}$ of f contained in Λ can be ϵ-shadowed by a point $y \in M$. The notion of pseudo orbits often appears in several methods of the modern theory of dynamical system ([9]). Moreover, the pseudo orbit shadowing property usually plays an important role in the investigation of stability theory and ergodic theory.

We say that Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. A diffeomorphism f is said to be *robustly transitive* if every diffeomorphism in a C^1 neighborhood of f has a dense orbit. A rotation map f_α on the unit circle $\mathbb{S}^1$ is transitive if α is irrational, but f_α is not robustly transitive. A closed invariant set Λ is said to be *locally maximal* if there is a compact neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$.

Received August 14, 2012; Accepted October 10, 2012.
2010 Mathematics Subject Classification: Primary 34D10, 37C20, 37C50.
Key words and phrases: C^1-generic, hyperbolic, shadowing, transitive set, robustly transitive set.

Definition 1.1. An invariant compact set Λ is robustly transitive (or $f|_\Lambda$ is robustly transitive) if there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that

(a) $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$,
(b) for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is transitive, where $\Lambda_g(U)$ is called the continuation of Λ.

In dimension one, there do not exist robustly transitive diffeomorphisms: the diffeomorphisms with finitely many hyperbolic periodic points are open and dense in $\mathrm{Diff}(\mathbb{S}^1)$. On the other hand, for two-dimensional diffeomorphisms, every robustly transitive set is a basic set; that is, it is transitive and periodic points are dense. In particular, every robustly transitive surface diffeomorphism is Anosov and the unique surface which supports such diffeomorphisms is the torus $\mathbb{T}^2$ (for more detail, see [8]). As we can see in [5], the robustly transitive sets in dimension 3 are generically partially hyperbolic. Moreover, Bonatti et al. ([2]) proved that every C^1-robustly transitive diffeomorphism on a finite dimensional manifold has dominated splitting. There are many non-hyperbolic transitive sets such as skew products, Derived from Anosov, and deformations of the time-τ map X_τ of the flow of a transitive Anosov vector field X.

A closed invariant set Λ is called *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$, and $0 < \lambda < 1$ such that $\|Df^n|_{E(x)}\| \le C\lambda^n$ and $\|Df^{-n}|_{F(f^n(x))}\| \le C\lambda^n$ for all $x \in \Lambda$ and $n \ge 0$. We say that f is Anosov if M is hyperbolic for f.

We say that a subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$ is *residual* if $\mathcal{R}$ contains the intersection of a countable family of open and dense subsets of $\mathrm{Diff}^1(M)$; in this case $\mathcal{R}$ is dense in $\mathrm{Diff}^1(M)$. A dynamic property P is said to be C^1-*generic* if P holds for all diffeomorphisms which belong to some residual subset of $\mathrm{Diff}^1(M)$. A set Λ is *a basic set* if Λ is locally maximal and $f|_\Lambda$ is transitive. It is easy to see that if Λ is hyperbolic basic set, then the periodic points are dense therein. Of course, every elementary set is a basic set.

In this paper, we prove that C^1-generically, if f has the shadowing property on a robustly transitive set $\Lambda \subset M$, then Λ is hyperbolic for f. More precisely, we prove the following theorem.

Theorem 1.2. *There exists a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{G}$ has shadowing property on a robustly set $\Lambda \subset M$, then Λ is hyperbolic for f.*

It is well known that if p is a hyperbolic periodic point of f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\},$$

and

$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. Let p, q be hyperbolic periodic points of f. We say that p and q are *homoclinically related*, and write $p \sim q$ if $W^s(p)$ intersects transversely $W^u(q)$, and $W^u(p)$ intersects transversely $W^s(q)$. It is clear that if $p \sim q$ then $\mathrm{index}(p) = \mathrm{index}(q)$, that is, $\dim W^s(p) = \dim W^s(q)$.

To prove our main theorem, we will use the following result due to Mañé([8]).

LEMMA 1.3. *Let $f|_\Lambda$ be robustly transitive. If there are a C^1- neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, any periodic point of g in $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is hyperbolic and have the same index, then there exists a C^1-neighborhood $\mathcal{V}(f) \subset \mathcal{U}(f)$ of f such that for any $g \in \mathcal{V}(f)$, $\Lambda_g(U)$ is hyperbolic.*

To prove Theorem 1.2, it is enough to show the following proposition if we apply Lemma 1.3.

PROPOSITION 1.4. *For C^1-generic f, if f has shadowing property on a robustly transitive set $\Lambda \subset M$, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, any periodic point of $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is hyperbolic and has the same index.*

2. Proof of Theorem 1.2

Let M be as before, and let $f \in \mathrm{Diff}(M)$. Let p be a hyperbolic point of a diffeomorphism f. For some $\epsilon(p) > 0$, we define the so-called local stable manifold $W^s_{\epsilon(p)}(p)$ and the local unstable manifold $W^u_{\epsilon(p)}(p)$ as

$$W^s_{\epsilon(p)}(p) = \{y \in M : d(f^n(p), f^n(y)) \le \epsilon(p) \;\; for \;\; n \ge 0\}$$

and

$$W^u_{\epsilon(p)}(p) = \{y \in M : d(f^n(p), f^n(y)) \le \epsilon(p) \;\; for \;\; n \le 0\}.$$

Note that $W^s_{\epsilon(p)}(p) \subset W^s(p)$ and $W^u_{\epsilon(p)}(p) \subset W^u(p)$.

To prove the proposition, we need some lemmas.

LEMMA 2.1. *Let Λ be a transitive set. Suppose that f has the shadowing property on Λ. Then for any hyperbolic periodic points $p, q \in \Lambda$,*

$$W^s(p) \cap W^u(q) \neq \emptyset \quad \text{and} \quad W^u(p) \cap W^s(p) \neq \emptyset.$$

Proof. In this proof, we will show that $W^u(p) \cap W^s(q) \neq \emptyset$. Since p and q are hyperbolic saddles, there are $\epsilon(p) > 0$ and $\epsilon(q) > 0$ such that both $W^s_{\epsilon(p)}(p)$ and $W^u_{\epsilon(q)}(q)$ are C^1-embedded disks, and such that if $d(f^n(x), f^n(p)) \leq \epsilon(p)$ for $n \geq 0$, then $x \in W^u_{\epsilon(q)}(q)$. Set $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$, and let $0 < \delta = \delta(\epsilon) < \epsilon$ be the number of the shadowing property of $f|_\Lambda$ with respect to ϵ. To simplify, we assume that $f(p) = p$ and $f(q) = q$. Since Λ is transitive, there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. We can choose $l_1 > 0$ and $l_2 > 0$ such that

$$d(p, f^{l_1}(x)) < \delta, \ \ \text{and} \ \ d(q, f^{l_2}(x)) < \delta.$$

Without loss of generality, we may assume that $l_2 > l_1$. Take $k > 0$ such that $l_2 = l_1 + k$. Then we get the finite δ-pseudo orbit

$$\xi\prime = \{p, f^{l_1}(x), f^{l_1+1}(x), \ldots, f^{l_2-1}(x), q\}.$$

Extend the finite δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ as follows;
Put

$$\begin{cases} x_i & = f^i(p) \quad \text{for } \ i \leq 0, \\ f^{l_1+i}(x) & = x_i \quad\quad \text{for } \ 1 \leq i < l_2, \\ f^{l_2}(x) & = x_{l_1+k} \\ f^{l_2+i}(x) & = x_{l_1+k+i} \ \ \text{for all } \ i \geq 0. \end{cases}$$

Then

$$\begin{aligned} \xi &= \{\ldots,\ p,\ p, f^{l_1}(x),\ f^{l_1+1}(x),\ \ldots,\ f^{l_2}(x),\ q,\ q,\ \ldots\} \\ &= \{\ldots, x_{-1}, x = (p), x_1, \ldots, x_{l_1+k-1},\ x_{l_1+k}, x_{l_1+k+1}, \ldots\} \end{aligned}$$

is a δ-pseudo orbit. It is clear that $\xi \subset \Lambda$. Since f has the shadowing property on Λ, there is a point $y \in M$ such that $d(f^i(y),\ x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Then we see that

$$y \in W^u_\epsilon(p) \ \ \text{and} \ \ f^{l_1+k}(y) \in W^s_\epsilon(q).$$

Therefore $y \in W^u(p) \cap W^s(q)$. Thus $W^u(p) \cap W^s(q) \neq \emptyset$. $\square$

LEMMA 2.2. (Kupka-Smale) *There is a residual set $\mathcal{G}_1 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_1$, every periodic point of f is hyperbolic, and all their invariant manifolds are transverse.*

End of the Proof of Theorem 1.2. Let $\mathcal{G}_1$ be the set of Kupka-Smale diffeomorphisms f in $\text{Diff}^1(M)$. It follows from Lemma 2.1 and $f \in \mathcal{G}_1$ that if f has shadowing property on Λ, then every periodic point in Λ has the same index.

Take a countable basis $\beta = \{U_n\}_{n\in\mathbb{N}}$ of M such that the union of two elements in β belongs to β. For each $U_n \in \beta$, we define $\mathcal{H}_n$ by the set of all diffeomorphism f has a C^1-neighborhood $\mathcal{U}(f)$ of f with the following properties: for any $g \in \mathcal{U}(f)$, if the maximal g-invariant set $\Lambda_g(U_n)$ in U_n, and g has two hyperbolic periodic points in $\Lambda_g(U_n)$ with different indices, then f has two hyperbolic periodic points in Λ with different indices. Then it is clear that $\mathcal{H}_n$ is open in $\text{Diff}(M)$ for each $n \in \mathbb{N}$.

Let $\mathcal{N}_n = \text{Diff}(M) \setminus \overline{\mathcal{H}_n}$. To show that $\mathcal{H}_n$ is dense in $\text{Diff}^1(M)$ for each $n \in \mathbb{N}$, we suppose that there is $f \notin \overline{\mathcal{N}_n}$. Then we can find a C^1-neighborhood $\mathcal{U}(f)$ of f such that for every $g \in \mathcal{U}(f)$, g has two hyperbolic periodic points $p_g, q_g \in \Lambda_g(U_n)$ with different indices. Since p_g, q_g are hyperbolic periodic orbits for g, by the stability theorem, f has two hyperbolic periodic points $p, q \in \Lambda$ with different indices. Thus $g \in \mathcal{H}_n$. This verifies that $\mathcal{H}_n \cup \mathcal{N}_n$ is C^1 dense in $\text{Diff}(M)$. Let $\mathcal{G}_2 = \bigcap_{n\in\mathbb{N}}(\mathcal{H}_n \cup \mathcal{N}_n)$. Then $\mathcal{G}_2$ is C^1-residual in $\text{Diff}(M)$.

Put $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$. Let $f \in \mathcal{G}$ and let $\Lambda \subset M$ be the robustly transitive set. Then there exists a basic element $U_{n_0} \in \beta$ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U_{n_0})$.

Suppose that Λ is not hyperbolic for f. Since Λ is the robustly transitive set, there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_{g(U)}}$ is transitive. Since Λ is nonhyperbolic, for $g \in \mathcal{U}(f)$, there are $p_g, q_g \in \Lambda_g(U) \cap P(g)$ with $\text{index}(p_g) \neq \text{index}(q_g)$. By the local stability theorem, there are uniquely $p, q \in \Lambda \cap P(f)$ such that $\text{index}(p) \neq \text{index}(q)$. By Lemmas 2.1 and 2.2, this is a contradiction to the fact that $f \in \mathcal{G}$. Thus any periodic point of $g \in \Lambda_g(U_{n_0})$ is hyperbolic and have the same index. Thus by Lemma 1.3, we see that $\Lambda_g(U_{n_0})$ is hyperbolic for g, and so Λ is hyperbolic for f. This completes the proof of Proposition 1.4. $\square$

Acknowledgments

This paper was supported by Daejeon Metropolitan Office of Education. It is a pleasure to thank Dr. Manseob Lee (Mokwon University) for useful discussion in composing this paper.

References

[1] C. Bonatti and L. J. Díaz. *Nonhyperbolic transitive diffeomorphisms*, Ann. Math. **143** (1996), 357-396.

[2] C. Bonatti, L. J. Díaz, and E. Pujals. *A C^1-generic dichotomy for diffeomorphsims: weak forms of hyperbolicity or infinitely many sinks or sources*. Ann. Math. **158** (2003), 355-418.

[3] C. Bonatti, L. J. Díaz, E. Pujals, and J. Rocha. *Robust transitivity and heterodimensional cycles.* Asterisque, **297** (2003), 187-222.

[4] L. J. Díaz. *Robust nonhyperbolic dynamics and heterodimensional cycles.* Ergod. Th. Dynam. Sys. **15** (1995), 291-315.

[5] L. J. Díaz, E. Pujals, and R. Ures. *Partial hyperbolicity and robust transitivity.* Acta Math. **183** (1999), 1-43.

[6] K. Lee, and M. Lee, *stably inverse shadowable transitive sets and dominated splitting*, Proc. Amer. Math. Soc. **140** (2012), 217-226.

[7] K. Lee, and X. Wen, *Shadowable chain transitive sets of C^1-generic diffeomorphisms*, Bull. Korean Math. Soc. **49** (2012), 21-28.

[8] R. Mañé, *An ergodic closing lemma*, Ann. Math. **116** (1982), 503-540.

[9] S. Y. Pilyugin, *Shadowing in Dynamical Systems*, Lect. Notes in Math.,Vol. **1706**, Springer, 1999.

[10] M. Shub, *Topologically transitive diffeomorphisms on T^4*, Lect. Notes in Math., Vol. **206**, Springer, 1971.

*

Daejeon Science High School
Daejeon 305-338, Republic of Korea
E-mail: kimdj623@hanmail.net